Optimizing the Incidences between Points and Arcs on a Circle

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Abstract: Given a set P of $2n+1$ points regularly spaced on a circle, a number π for pairwise distinct points and a number α for pairwise distinct and fixed length arcs incident to points in P, the sum of incidences between α arcs and π points, is optimized by contiguously assigning both arcs and points. An extension to negative incidences by considering ± 1 weights on points is provided. Optimizing a special case of a bilinear form (Hardy, Littlewood and Pólya's theorem) as well as Circulant$\times$ anti-Monge QAP directly follow.

Key-words: bilinear form, QAP

(Résumé : tsvp)

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Optimisation du nombre d'incidences entre des points et des arcs sur un cercle

Résumé : Étant donné un ensemble P de $2n+1$ points régulièrement espacés sur un cercle, optimiser la somme des incidences entre π points distincts et α arcs de longueur fixe et adjacents aux points de P, revient à affecter de manière contiguë à la fois les π

points et les α arcs. Ce résultat s'étend aux incidences signées en considérant des poids ± 1 aux points. L'optimisation d'une certaine forme bilinéaire (théorème de Hardy, Littlewood et Pólya) ainsi que l'affectation quadratique dans le cas Circulant$\times$ anti-Monge en d'ecoule directement.

Mots-clé : forme bilinéaire, QAP

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1 Introduction

Given a point and arcs on a circle, let define the incidences in this point as the number of arcs covering it; the dual notion, i.e. given an arc and points on a circle, the *length* of the arc is the number of points covered by the arc minus 1 to be conformal with standard notion of length.

In a note on a certain bilinear form [eW94], Çela and Woeginger prove that given a set P of $2n+1$ points regularly spaced on a circle, the way to place π pairwise distinct points and α pairwise distinct and fixed

length arcs incident to points in P such that the total number of incidences is maximized, amounts to contiguously assign both arcs and points.

The purpose of this note is to prove that the same holds if minimization replaces maximization.

First, let us recall notations and definitions We provide the circle with an orientation, say clockwise, and reference arcs from their origin w.r.t. orientation (*left* endpoint in a unfolding of the circle); blocks are a sequence of arcs adjacent through their origin.

- fixed arc length: $\lambda \geq 2$,
- arc: $a_l = [l, l+\lambda-1]$,
- block (of k adjacent arcs): $b_l^k = [l, l+\lambda+k-2]$, where k is named for obvious reasons, the cardinality of the block $|b_l^k| = k$,
- solution: $\mathcal{S} = (\mathcal{P}, \mathcal{B})$ where $|\mathcal{P}| = \pi$ and $|\mathcal{B}| = \sum_{b\in\mathcal{B}} |b| = \alpha$,
- incidences at a given point: I_l = number of arcs covering l;
- optimal solution: $\mathcal{S}^* = \min_{\mathcal{S}=(\mathcal{P},\mathcal{B})} \sum_{p\in\mathcal{P}} I_p$.

2 The result

Call $\mathrm{IPA}_{2n+1}(\pi, \alpha, \lambda)$ the minimization of Incidences between Points and Arcs as defined above; this problem is self-dual in the following sense.

1. consider an optimum $(\mathcal{P}^*, \mathcal{B}^*)$ to $\mathrm{IPA}_{2n+1}(\pi, \alpha, \lambda)$ and replace every point $p \in \mathcal{P}^*$ by an arc of length λ with p as its end and replaces every arc in $\mathcal{B}^*$ by its origin, then the new configuration is optimal for $\mathrm{IPA}(\alpha, \pi, \lambda)$ since blocks become intervals and intervals become blocks,

2. consider an optimum $(\mathcal{P}^*, \mathcal{B}^*)$ to $\mathrm{IPA}_{2n+1}(\pi, \alpha, \lambda)$ and replace $\mathcal{P}^*$ by $P \setminus \mathcal{P}^*$ and every arc in $\mathcal{B}^*$ by its complement w.r.t. the circle, then the new configuration is optimal for $\mathrm{IPA}_{2n+1}(2n+1-\pi, \alpha, 2n+1-\lambda)$ since intervals and blocks remain intervals and blocks.

Therefore, we may restrict to $\lambda \leq n$ ($\alpha \leq n$ since multiple arcs are forbidden). Observe next that in case of $\lambda = 1$, the problem has a trivial solution fulfilling our goal.

Lemma 2.1 *There exists an optimum solution without blocks of cardinality 1.*

Proof.

Let suppose the contrary, i.e. in an optimal solution $\mathcal{S}^0$, a block b_l^1 is alone (non adjacent to anyone else); it means that no other block in the solution starts in $l-1, l+1$ or ends in $l+\lambda-1, l+\lambda+1$. The idea is to transfer weight from largest incidences to smallest by circularly shifting this arc until it merges with a block in $\mathcal{S}^0$.

Since no multiple arcs is allowed, a case study leads to $I_{l+1}^0 = I_l^0 - \epsilon$ and $I_{l+\lambda}^0 = I_{l+\lambda-1}^0 - \epsilon$, with $\epsilon = 0$ or 1 for both endpoints of arc. Of no use for the proof, we may notice $I_{l-1}^0 = I_l^0 - \epsilon$ and $I_{l+\lambda-2}^0 = I_{l+\lambda-1}^0 - \epsilon$ as well.

Let assume w.l.o.g. that $I_l^0 \leq I_{l+\lambda-1}^0$ and shift the arc, one step clockwise. the new incidences become $I_l^1 = I_l^0 - 1, I_{l+\lambda}^1 = I_{l+\lambda}^0 + 1$ while all others remain unchanged; then, examine every different incidences at l:

- $I_l^0 = I_{l+\lambda}^0 + 1$; clearly the incidences remain the same, $I_l^1 = I_{l+\lambda}^0$ and $I_{l+\lambda}^1 = I_l^0$ and our assumption remains true as well, $I_{l+1}^1 = I_{l+1}^0 \leq I_l^0 = I_{l+\lambda}^0 + 1 = I_{l+\lambda}^1$
- $I_l^0 \leq I_{l+\lambda}^0$, then $I_{l+1}^1 = I_{l+1}^0 \leq I_l^0 < I_{l+\lambda}^0 + 1 = I_{l+\lambda}^1$ meanwhile if $l+\lambda \in \mathcal{P}^0$ then $l \in \mathcal{P}^0$ and the number of incidences cannot increase in this move.
- $I_l^0 > I_{l+\lambda}^0 + 1$; then from case study $I_{l+\lambda-1}^0 = I_{l+\lambda}^0 + \epsilon < I_l^0 - 1 + \epsilon < I_l^0$ a contradiction.

Recursively applying above shift leads to a sequence of optimal solutions $\mathcal{S}^i, i = 0, \ldots q$ until the arc merges another block in $\mathcal{S}^0$. Processing all single blocks this way, achieves the result. □

Observation 2.2 *For a block b_l^k that does not overlap with any other, incidences increase from l upto $\min(k, \lambda+1)$, remain constant and then decrease upto right endpoint of block. If the block self overlaps, left and right overlapped incidences are shifted by the overlapping length.*

Now we generalize lemma 2.1 to the blocks themselves in order to reach the main goal.

Lemma 2.3 *There exists an optimum solution with only 1 block.*

Proof.

Let an optimal solution $\mathcal{S}^0$, $b_l^k = [l, r = l + k + \lambda - 2]$ be a block such that $I_l^0 \leq I_r^0$ and shift it one step clockwise; by definition, no other block in $\mathcal{S}^0$ starts in interval $[l-1, r+1-\lambda = l-1+k]$ or ends in interval $[l-1+\lambda, r+1 = l-1+k+\lambda]$ (or else a multiple arc would exist). Let $m = \min(l-1+\lambda, l-1+k)$ and $M = \max(l-1+\lambda, l-1+k)$, then from $I_l^0 \leq I_r^0$ and observation 2.2, *left* overlapping length is less or equal than *right* overlapping length; in other words, there exist $L \in [l, m]$ and $R \in [M, r]$ such that

$$\begin{aligned}
I_i^0 &= I_{l-1}^0, \quad \text{for all } i \in [l, L] \\
I_i^0 &= I_{r+1}^0, \quad \text{for all } i \in [R, r] \\
L - l &\leq r - R \\
I_{i+1}^1 &= I_i^0, \quad \text{for all } i \in [L+1, R-2] \\
I_i^1 &= I_i^0 - 1, \quad \text{for all } i \in [l, L+1] \\
I_i^1 &= I_i^0 + 1, \quad \text{for all } i \in [R, r+1]
\end{aligned}$$

From equalities in the midpart of the block, we may restrict study of incidences in $[l, L+1]$ and $[R, r+1]$:

- $I_l^0 = I_r^0$; clearly the incidences remain the same in the following sense :

$$\sum_{i=l}^{l+j} I_i^0 + \sum_{i=r+1-j}^{r+1} I_i^0 = \sum_{i=l}^{l+j} I_i^1 + \sum_{i=r+1-j}^{r+1} I_i^1, \quad \text{for all } 0 \leq j \leq L+1-l = r+1-R$$

  Therefore, upto rebalancing assignment of points in $\mathcal{P}^0$ from *right* part to *left* part, we found another optimal solution $\mathcal{S}^1 = (\mathcal{P}^1, \mathcal{B}^1)$ such that assumption remains true (since $I_{l+1}^1 = I_{r+1}^1 - 2 < I_{r+1}^1$),

- $I_l^0 < I_r^0$; it should exist a $p \in \mathcal{P}^0$ such that $p \in [R, r+1]$ or else the shifted version is better than $\mathcal{S}^0$, a contradiction. So, $[l, l+r-R] \in \mathcal{P}^0$ as well so that

$$\begin{aligned}
I_{l+r-R+1}^0 + I_{r+1}^0 &= I_{l+r-R+1}^1 + I_{r+1}^1 \\
\sum_{i=l}^{l+r-R} I_i^0 + \sum_{i=R}^{r+1} I_i^0 &= \sum_{i=l}^{l+r-R} I_i^1 + \sum_{i=R}^{r+1} I_i^1
\end{aligned}$$

  it means that the related assignment $\mathcal{S}^1 = (\mathcal{P}^0, \mathcal{B}^1)$ remains optimal. Since assumption still holds $I_{l+1}^1 = I_l^0 < I_r^0 = I_{r+1}^0 = I_{r+1}^1 - 1$, we are done.

Recursively apply above shift until the block merges another block in $\mathcal{S}^0$, and process all blocks w.r.t. $I_l \leq I_r$ ordering to reach an optimal solution with 1 block only. □

Theorem 2.4 *Let P be a set of $2n+1$ points regularly spaced on a circle, an optimal way to minimize incidences between π pairwise distinct points and α pairwise distinct arcs of length λ, incident to points in P, consists in one block of α adjacent arcs and π adjacent points on the circle.*

Proof.

From an optimal solution with only 1 block, lemma 2.3 together with bitonic property 2.2 allows to select π points as follows :

1. block does not self-overlap; assign $p = \min(\pi, 2n+1-(\lambda+\alpha-2))$ contiguous points in the complement of *optimal* block b_l^k; then assign remaining $\pi - p$ points alternately from the left and right endpoints of block,

2. block self-overlaps; assign π points alternately from the left and right endpoints of block towards its midpart.

$\square$

Observation 2.5 *The restriction to odd number of points arose for the sake of original form of cited theorems but was irrelevant on the incidences reasoning; so, the subscript under IPA refers to previous work only.*

3 Application to HLP's theorem on a bilinear form

Let us give a short proof of a theorem by Hardy, Littlewood and Pólya (1926) about the minimum on a certain bilinear form and recently refined (1994) by Çela and Woeginger.

Definition 3.1 *A function $f : [-2n, 2n] \longmapsto \mathbb{R}$ is called weakly symmetric increasing if it fulfills the symmetric, non decreasing and dominant properties*

$$\begin{aligned} f(-i) &= f(i), \quad \text{for all } 1 \le i \le 2n \text{ (SYM)} \\ f(i-1) &\le f(i), \quad \text{for all } 1 \le i \le n \text{ (INC)} \\ f(i) &\le f(2n+1-i), \quad \text{for all } 1 \le i \le n \text{ (DOM)} \end{aligned}$$

Definition 3.2 *The permutation $\rho^+ \in S_{2n+1}$ is defined as*

$$\begin{aligned} \rho(2p) &= n+1+p, \quad \text{for all } p \le n \\ \rho(2p+1) &= n+1-p, \quad \text{for all } p \le n \end{aligned}$$

For a sequence (vector resp.) $x \in \mathbb{R}^{2n+1}$ and a permutation $\phi \in S_{2n+1}$, we note x_ϕ the permuted sequence (vector resp.). We denote x^+ the reordering of x such that the non increasing values of x are alternately distributed around center location as in ρ^+ (right then left according to the even/odd choice of sign of p in ρ^+ definition).

Theorem 3.1 *(Çela and Woeginger [eW94]) Let f be a non negative weakly increasing function and non negative $x, y \in \mathbb{R}_+^{2n+1}$. Then the bilinear form $\sum_{r=-n}^{n} \sum_{s=-n}^{n} f(r-s) x_r y_s$ attains its minimum over all arrangement of x, y, for x^+ and y^+ in S_{2n+1}; in compact form*

$$\begin{aligned} (\phi^*, \psi^*) &= \arg\min_{\phi,\psi} \sum_{r=-n}^{n} \sum_{s=-n}^{n} f(r-s) x_{\phi(r)} y_{\psi(s)} \\ & \text{s.t.} \quad \begin{cases} x_{\phi^*} = x^+ \\ y_{\psi^*} = y^+ \end{cases} \end{aligned}$$

First, observe that non negative weakly increasing functions form a cone so the proof amounts to prove it for the extremal rays.

Lemma 3.2 *(Çela and Woeginger [eW94]) Assume that all elements of the sequences* $x, y \in \mathbb{R}^{2n+1}$ *are 0 or 1 and that for some integer* $1 \leq d \leq n, f : [-2n, 2n] \longmapsto \mathbb{R}$ *is defined by:*

$$f_i = \begin{cases} -1 & i = \pm(n+d) \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

Then the bilinear form attains its minimum for x^+, y^+.

Proof.

Considering the special structure of f, fixing $x_r = 1$ for some $-n \leq r \leq n$ then minimizing the form is equivalent to maximizing subexpressions

$$y_{r+n+d} + y_{r-n-d}$$

Thinking of the y_s as a circular sequence of $2n+1$ equidistant points on a circle, and of the x_r as intervals that covers $2(n-d+1)$ consecutive points on the circle establishes the correspondence between minimization of above bilinear form and maximization of incidences between points and arcs; it stems from the perfect symmetry around each r so that covering both endpoints of arc y_{r+n+d}, y_{r-n-d} is equivalent to covering all intermediate points. The authors, in their original proof, did not recourse to this correspondence since a direct proof is quite easy. □

Theorem 3.3 *(Çela and Woeginger [eW94]) Assume that all elements of the sequences* $x, y \in \mathbb{R}^{2n+1}$ *are 0 or 1 and that for some integer* $1 \leq d \leq n, f : [-2n, 2n] \longmapsto \mathbb{R}$ *is defined by:*

$$f_i = \begin{cases} 0 & n-d+1 \leq i \leq n+d \textit{ or } -n-d \leq i \leq -n+d-1 \\ 1 & \textit{otherwise} \end{cases} \tag{2}$$

Then the bilinear form attains its minimum for x^+, y^+.

Proof.

For some fixed $x_r = 1, -n \leq r \leq n$, the bilinear form rewrites

$$\begin{array}{ll} \displaystyle\sum_{s=-n}^{r+n-d} y_s + \sum_{s=r+n+d}^{n} y_s & \text{if } \ -n \leq r \leq -d-1 \\ \displaystyle\sum_{s=r-n+d}^{r+n-d} y_s & \text{if } \ -d \leq r \leq d \\ \displaystyle\sum_{s=-n}^{r-n-d-1} y_s + \sum_{s=r-n+d}^{n} y_s & \text{if } \ d+1 \leq r \leq n \end{array}$$

and the correspondence established by the authors about minimizing the bilinear form and minimizing the number of incidences between points covering $2(n-d+1)-1$ consecutive points and intervals becomes clear. □

4 ± 1 weighted incidences

Above result considers a $0-1$ labelling of points on the circle; intuitively, it suggests that the result still holds with a labelling of points by -1 as well.

Theorem 4.1 *Let P be a set of $0, \pm 1$ weighted points regularly spaced on a circle, an optimal way to minimize incidences between π(μ resp.) pairwise distinct, $+1(-1)$resp.)-weighted points and α pairwise distinct arcs of length λ consists in one block of α adjacent arcs and π(μ resp.) adjacent $+1(-1)$resp.)-weighted points on the circle.*

Proof.

(i) **arcs are blocked.** Considering 0s and +1s only, lemma 2.1 avoids blocks of cardinality 1; the maximization counterpart over 0s and -1s follows from the corresponding lemma. Observation 2.2 on non overlapping arcs together with a symmetry argument leads for a block b^k, to assign -1 to the constant highest incidences and the $+1$ to lowest incidences (symmetrically starting from both endpoints). Let suppose b^k contains $2\pi^k + \epsilon^k$ of $+1$-weighted points and μ^k -1-weighted points, the best contribution of this block to the overall incidences is $(\pi^k + \epsilon^k)(\pi^k + 1) - \mu^k(\lambda + 1)$ where ϵ^k accounts for a possibly odd number of points.

(ii) **an optimal solution is achieved through a single block.** Gathering all the blocks into the global optimization of incidences we get for the non self-overlapping case

$$
\begin{aligned}
&\min \sum_{b^k} (\pi^k + \epsilon^k)(\pi^k + 1) - \mu^k(\lambda + 1) \\
&\sum_{b^k} (2\pi^k + \epsilon^k) = \pi \qquad (\text{\#+1-POINTS}) \\
&\sum_{b^k} \mu^k = \mu \qquad (\text{\#-1-POINTS}) \\
&\sum_{b^k} k + \lambda + 1 = |P| \qquad (\text{PERIMETER}) \\
&\sum_{b^k} k = \alpha \qquad (\text{\#ARCS})
\end{aligned}
$$

Considering the lagrangian dual with p(m resp.) as multipliers for $+1(-1$ resp.)-weighted points,

$$
\mathcal{L}(p, m) = \min \sum_{b^k} (\pi^k + \epsilon^k)(\pi^k + 1) - \mu^k(\lambda + 1) + p \sum_{b^k} (2\pi^k + \epsilon^k) + m \sum_{b^k} \mu^k - m\mu - p\pi
$$

we find that p should decrease $(2\pi^k + \epsilon^k + 1 + 2p = 0)$ while m should increase $(-(\lambda+1) + m = 0)$; back into the dual lagrangian, we arrive, by discarding *constants* terms in ϵ^k and $(\lambda + 1)\mu$ offset, at $\max \sum_{b^k} (\pi^k - \frac{\pi - \epsilon^k}{2})^2$ which requires π^k to be 0 for every block but one.

Notice that overlapping blocks do not change the incidences count, hence the result.

5 Application to the QAP

Let incidences on a circle be set in a matrix notation. First, choose an origin and an orientation for points on the circle and associate with all possible arcs, a matrix Λ of 0s and 1s where 1s correspond to the points covered by an arc; assuming that arcs do not overlap, we observe that Λ is generated through a circular shift of an incidence vector of a generic arc. Then, let A denotes the vector of reference points associated to a set of α arcs w.r.t. orientation on the circle; we implicitly select the left endpoint as the reference point but any other point is suited, provided the Λ matrix is defined accordingly. Last, let P be a vector of weighted incidence points w.r.t. above orientation. Optimization of incidences between points and arcs $\mathrm{IPA}(\pi, \alpha, \lambda)$ set in matrix notation is nothing else than

$$\mathrm{IPA}(P, A, \Lambda) \quad = \quad < \mathrm{diag}(A)\Lambda P, E >$$

where E is the all 1s vector and the inner product abbreviates the sum over all points.

Using Hadamard (Schur [HJ91]) product and Kronecker definitions, we have on the one hand, for incidences

$$\mathrm{diag}(A)\Lambda P \quad = \quad (\Lambda \circ (A \otimes P^t))E$$

and on the other hand, for the quadratic assignment problem asociated to matrices F and D

$$\mathrm{QAP}(F, D) \quad = \quad < (F \circ D)E, E >$$

In other words

$$\mathrm{IPA}(P, A, \Lambda) \quad = \quad \mathrm{QAP}(\Lambda, A \otimes P^t)$$

Let $A = (1, 1, 0, 0)$ and $P = (1, 1, 0, 0, -1, -1)$ be an instance of the incidences, we may notice that it induces an anti-Monge property on $A \otimes P^t$ apart from the circulant property of Λ matrix. More precisely, the scope of the result on Toeplitz $\times$ anti-Monge QAP by Çela et al. [BeRW98, Çel98] extends to circulant $\times$ anti-Monge QAP without non negative constraint on anti-Monge matrix.

6 Concluding remarks

In this paper, we extend the optimization of non negative incidences of points and arcs on a circle [eW94], to the negative weighted case; it allows to rewrite a theorem by Hardy, Littlewood and Pólya (1926) about the minimum on a certain bilinear form on signed number as well (probably known from these authors but useless at that time). More interestingly, it removes the same restriction on non negativity of anti-Monge matrices in circulant $\times$ anti-Monge QAP; with a little effort, it may be carried over the more general Toeplitz-benevolent $\times$ anti-Monge QAP.

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